Topology

M. Math. I

Semestral Examination

Instructions: All questions carry equal marks.

1. Let $\{A_\alpha\}$ be a collection of subsets of a topological space X such that for every $x \in X$, there exists an open set U_x containing x which intersects only finitely many A_α's. Then prove that
$$\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$$
Give an example of a topological space Y and subsets $\{B_\alpha\}$ for which this equality does not hold.

2. Define a quotient map beween two topological spaces. Define an equivalence relation on the unit sphere $S^n \subset \mathbb{R}^{n+1}$ by $x \sim y$ iff $x = \pm y$. Prove that the set of equivalence classes with the quotient topology is normal.

3. Let X be topological space and x_0 be a point in X. Define the fundamental group $\pi_1(X, x_0)$. If X is path connected and if x_1 is any other point of X, then prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

4. Define a covering map. Let $p : E \to B$ be a covering map. If B is connected and if $p^{-1}(b_0)$ has cardinality n for some point $b_0 \in B$, then prove that $p^{-1}(b)$ has cardinality n for all points $b \in B$.

5. Let $p : E \to B$ be a covering map, with E path connected. If the fundamental group of B is trivial, then prove that p is a homeomorphism.

6. Let $N = (0, 0, 1)$ denote the *north pole* of the unit sphere $S^2 \subset \mathbb{R}^3$. Prove that any loop at N is homotopic to a loop that misses the *south pole* $S = (0, 0, -1)$. Hence, prove that the fundamental group of S^2 is trivial.